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PULSATILE FLOW IN HEATED POROUS CHANNEL

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Abstract—The combined forced and free convection flow through a porous channel when a pulsatile pressure is applied across its ends is discussed. It is assumed that the ratio of the width of the channel to the length (δ) is small. Even in this physically realistic situation of a finite channel, the transverse velocity remains undisturbed and acquires its suction or blowing value at the channel walls. A salient feature of the investigation is the presence of steady streaming component in the higher approximations due to nonlinearity in the viscous dissipation heat. When the channel is horizontal, the axial velocity distribution obtained is exact and independent of δ. But the pressure and temperature distributions are exact and δ-dependent. On the other hand, if the channel is vertical, only the pressure distribution is obtained exactly. It is independent of the transverse coordinate but varies linearly with the axial coordinate. The velocity and temperature are obtained approximately to order $O(\delta)$. Also the shear stress and heat flux at the walls are discussed quantitatively.

NOMENCLATURE

When written together, primed quantities are dimensional and unprimed quantities are dimensionless.

- C_0, constant defined in (2);
- C_p, heat capacity of fluid;
- $\bar{E}c$, modified Eckert number;
- Gr, Grashof number;
- g, gravitational acceleration;
- h, width of channel;
- K, constant defined in (2);
- k, thermal conductivity of fluid;
- L, representative length of channel;
- m, wall temperature ratio;
- p', P, pressure of fluid in channel;
- Pr, Prandtl number;
- P'_∞, pressure of fluid at infinity;
- R, Reynolds number;
- T', temperature of fluid channel;
- T_∞, temperature of fluid at infinity;
- T_0, temperature of lower wall;
- T_1, temperature of upper wall;
- t', t, time;
- u', u, longitudinal velocity;
- v', v, axial velocity;
- V_0, suction velocity;
- x, x', longitudinal coordinate;
- y, y', axial coordinate.

Greek symbols

- α, constant defined in (9);
- β, coefficient of volume expansion;
- δ, dimensionless parameter defined in (1);
- Θ, dimensionless temperature of the fluid;
- ρ', density of fluid in channel;
- ρ_∞, density of fluid at infinity;
- σ, frequency parameter;
- ω, frequency of pulsatile pressure.

1. INTRODUCTION

THE PROBLEM of determining the flow through a porous channel driven by a pulsatile pressure gradient is a fundamental one with obvious applications in physiology and engineering. Berman [1] and Wang [2] have studied the velocity distribution and shear stresses in an infinite channel while Radhakrishnamacharya and Maiti [3] discussed the heat transfer in the same configuration. A major application of the work is to the understanding of the dialysis of blood in artificial kidneys (see, e.g., [4]). Blood is taken to be a homogeneous and Newtonian fluid which is a good approximation when the shear rate exceeds 100 s^{-1} and in channels of depth greater than 100 μm [5].

The major concern of this work is two-fold. First of all the idealization that the channel is infinite is relaxed. The depth of the channel (h) will be assumed small compared with the length (KL) so that the parameter

$$\delta = \frac{h}{L} < 1. \tag{1}$$

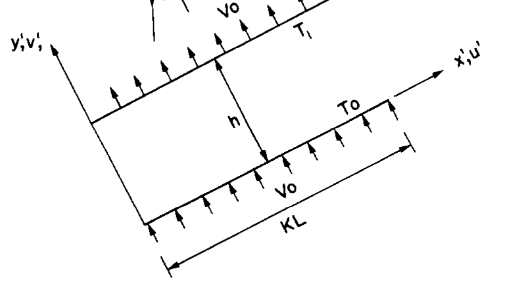

FIG. 1. Coordinate system.

In which case a pulsatile pressure applied across the ends of the channel is now the driving force. Secondly we assume that the difference in temperature between the walls and that of the undisturbed fluid is large enough for free convection currents to flow. The last condition may prevail in practice and therefore is physically important. The procedure adopted below is as follows.

In section 2 a non-dimensional form of the governing equations are deduced. Approximate solutions to these equations are given in section 3 on the assumption that δ is small. The shear stresses and the heat flux are discussed in sections 4 and 5 respectively.

2. FORMULATION

The physical problem consists of a porous channel inclined at angle φ to the horizontal (Fig. 1). A pulsatile pressure is applied across the ends of the channel and is given by

$$p'(KL, y', t') - p(0, y', t') = C_0(1 + \varepsilon \cos \omega t'), \quad 0 \leqslant x' \leqslant KL. \tag{2}$$

The plate at $y' = 0$ is maintained at temperature T_0 while fluid is blown into it with velocity V_0. That at $y' = h$ is kept at temperature T_1 and fluid sucked from it with velocity V_0. All the fluid properties are assumed constant except the density in the buoyancy force terms which varies according to the law

$$\rho_\infty - \rho' = \rho_\infty \beta(T' - T_\infty). \tag{3}$$

This is the Boussinesq approximation in which β represents the coefficient of volume expansion. Subscript ∞ refers to conditions in the undisturbed fluid.

Under this condition the equations of continuity, momentum and energy become

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{4}$$

$$\rho_\infty\left(\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p'}{\partial x'} + \mu\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) - \rho' g \sin\varphi, \tag{5}$$

$$\rho_\infty\left(\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial p'}{\partial y'} + \mu\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) - \rho' g \cos\varphi, \tag{6}$$

$$\rho_\infty C_p\left(\frac{\partial T'}{\partial t'} + u'\frac{\partial T'}{\partial x'} + v'\frac{\partial T'}{\partial y'}\right) = k\left(\frac{\partial^2 T'}{\partial x'^2} + \frac{\partial^2 T'}{\partial y'^2}\right) + 2\mu\left[\left(\frac{\partial u'}{\partial x'}\right)^2 + \left(\frac{\partial v'}{\partial y'}\right)^2\right] + \mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)^2, \tag{7}$$

subject to

$$\begin{aligned} &u' = 0, \quad v' = V_0, \quad T' = T_0 \quad \text{at } y' = 0, \\ &u' = 0, \quad v' = V_0, \quad T' = T_1 \quad \text{at } y' = h. \end{aligned} \tag{8}$$

All the primed quantities are dimensional.

To facilitate analysis we now introduce the following non-dimensional quantities:

$$t = \omega t', \quad x = x'/L, \quad y = y'/h,$$
$$(u, v) = (u', v')/V_0, \quad P = (p' - p'_\infty)h/\mu V_0,$$
$$\Theta = \frac{T' - T_\infty}{T_0 - T_\infty}, \quad \alpha = hC_0/\mu V_0, \quad m = \frac{T_1 - T_\infty}{T_0 - T_\infty}, \tag{9}$$
$$R = V_0 h/\nu, \quad Pr = \mu C_p/k, \quad Gr = \frac{g\beta h^2(T_0 - T_\infty)}{\nu V_0},$$
$$\sigma = \omega h^2/\nu, \quad \bar{E}c = \frac{V_0^2 L}{hC_p(T_0 - T_\infty)},$$

Here m is the wall temperature ratio. The non-dimensional quantities R (Reynolds number) and Pr (Prandtl number) are of order $O(1)$ for the problem under investigation. Gr, the Grashof number or free convection parameter should be of order $O(1)$ for appreciable free convection currents to flow. The frequency parameter will be assumed to be of order $O(1)$. In the major blood vessels of the coronary circulation of man, σ is usually of order $O(0.7 \times 2\pi R)$. The Eckert number $Ec = V_0^2/C_p(T_0 - T_\infty)$ is much less than unity for all incompressible flows. Thus we have introduced a modified Eckert number $\bar{E}c$ obtained from Ec scaled by δ^{-1}, so that $\bar{E}c$ will be of order $O(1)$ since $\delta \ll 1$. Thus we have

$$\delta\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{10}$$

$$\sigma\frac{\partial u}{\partial t} + R\left(\delta u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\delta\frac{\partial P}{\partial x} + \delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Gr\,\Theta\sin\varphi, \tag{11}$$

$$\sigma\frac{\partial v}{\partial t} + R\left(\delta u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \delta^2\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + Gr\,\Theta\cos\varphi, \tag{12}$$

$$Pr\,\sigma\frac{\partial\Theta}{\partial t} + Pr\,R\left(\delta u\frac{\partial\Theta}{\partial x} + v\frac{\partial\Theta}{\partial y}\right) = \delta^2\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + 2Pr\,\bar{E}c\,\delta\left[\delta^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + Pr\,\bar{E}c\,\delta\left(\frac{\partial u}{\partial y} + \delta\frac{\partial v}{\partial x}\right)^2; \tag{13}$$

subject to

$$u = 0, \quad v = 1, \quad \Theta = 1 \quad \text{at } y = 0,$$
$$u = 0, \quad v = 1, \quad \Theta = m \quad \text{at } y = 1,$$

$$P(K,y,t) - P(0,y,t) = \alpha(1 + \varepsilon\cos t). \quad (14)$$

The problem is now reduced to mathematical terms. We want to solve (10)–(13) subject to (14). This problem is non-linear, coupled and not easily amenable to analytical treatment. Since all parameters in equations (10)–(13) are of order $O(1)$ except δ, asymptotic analysis is in order and this will be discussed in the next section. Before then, we note that in (7) we have retained the viscous dissipative terms that are usually neglected in free convection problems involving low velocities. But as shown by Gebhart [6] and Gebhart and Mollendorf [7], viscous dissipative effects play an important role in natural convection flow fields of extreme size, or extremely low temperature or in high gravity. These are various situations prevalent in physiology and engineering. We have also considered a channel of arbitrary inclination φ. In all quoted works in section 1, φ is taken as zero. However other values of φ are possible in practical applications, especially $\varphi = \pi/2$. For versatility φ will be left arbitrary in the analysis and various limits discussed as special cases.

3. APPROXIMATE SOLUTIONS

In order to make any headway with the analysis we shall assume that δ is small and all other parameters are of order $O(1)$ in equations (10)–(13). We then expand the axial velocity and temperature in the form

$$u = u^{(0)} + \delta u^{(1)} + O(\delta^2), \quad \text{etc.} \quad (15)$$

while the transverse velocity is approximated by

$$v = 1 + \delta v^{(1)} + O(\delta^2), \quad (16)$$

and the pressure expressed as

$$P = \frac{1}{\delta}P^{(0)} + P^{(1)} + O(\delta). \quad (17)$$

When we substitute (15)–(17) into (10)–(14) we obtain the sequence of approximations:

$$\sigma\frac{\partial u^{(0)}}{\partial t} + R\frac{\partial u^{(0)}}{\partial y} = -\frac{\partial P^{(0)}}{\partial x} + \frac{\partial^2 u^{(0)}}{\partial y^2} + Gr\,\Theta^{(0)}\sin\varphi, \quad (18)$$

$$0 = -\frac{\partial P^{(1)}}{\partial y} + Gr\,\Theta^{(0)}\cos\varphi, \quad (19)$$

$$Pr\,\sigma\frac{\partial\Theta^{(0)}}{\partial t} + Pr\,R\frac{\partial\Theta^{(0)}}{\partial y} = \frac{\partial^2\Theta^{(0)}}{\partial y^2}, \quad (20)$$

$$u^{(0)} = 0, \quad \Theta^{(0)} = 1 \quad \text{at } y = 0, \quad (21)$$

$$u^{(0)} = 0, \quad \Theta^{(0)} = m \quad \text{at } y = 1, \quad (22)$$

$$P^{(0)}(K,y,t) - P^{(0)}(0,y,t) = \alpha(1 + \varepsilon\cos t); \quad (23)$$

$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \quad (24)$$

$$\sigma\frac{\partial u^{(1)}}{\partial t} + R\,u^{(0)}\frac{\partial u^{(0)}}{\partial x} + R\left(\frac{\partial u^{(1)}}{\partial y} + v^{(1)}\frac{\partial u^{(0)}}{\partial y}\right) = -\frac{\partial P^{(1)}}{\partial x} + \frac{\partial^2 u^{(0)}}{\partial x^2} + \frac{\partial^2 u^{(1)}}{\partial y^2} + Gr\,\Theta^{(1)}\sin\varphi, \quad (25)$$

$$\sigma\frac{\partial v^{(1)}}{\partial t} + R\frac{\partial v^{(1)}}{\partial y} = -\frac{\partial P^{(1)}}{\partial y} + \frac{\partial^2 v^{(1)}}{\partial y^2} + Gr\Theta^{(1)}\cos\varphi, \quad (26)$$

$$Pr\,\sigma\frac{\partial\Theta^{(1)}}{\partial t} + Pr\,R\,u^{(0)}\frac{\partial\Theta^{(0)}}{\partial x} + Pr\,R\left(\frac{\partial\Theta^{(1)}}{\partial y} + \frac{\partial\Theta^{(0)}}{\partial y}v^{(1)}\right) = \frac{\partial^2\Theta^{(1)}}{\partial y^2} + Pr\,\bar{E}c\left(\frac{\partial u^{(0)}}{\partial y}\right)^2, \quad (27)$$

$$u^{(1)} = v^{(1)} = \Theta^{(1)} = 0 \quad \text{at } y = 0, \quad (28)$$

$$u^{(1)} = v^{(1)} = \Theta^{(1)} = 0 \quad \text{at } y = 1, \quad (29)$$

$$P^{(1)}(K,y,t) - P^{(1)}(0,y,t) = 0, \quad \text{etc.} \quad (30)$$

3.1. *Order* O(1) *solution*

The solution of equation (20) subject to (21) and (22) is easily deduced as

$$\Theta^{(0)} = 1 + \frac{m-1}{e^{Pr\,R} - 1}(e^{Pr\,Ry} - 1). \quad (31)$$

By virtue of (31), (19) becomes

$$P^{(0)} = f^{(0)}(x,t), \quad (32)$$

in which case (18) reduces to

$$\sigma\frac{\partial u^{(0)}}{\partial t} + R\frac{\partial u^{(0)}}{\partial y} = -\frac{\partial f^{(0)}}{\partial x} + \frac{\partial^2 u^{(0)}}{\partial y^2} + Gr\,\Theta^{(0)}\sin\varphi.$$

Putting

$$u^{(0)} = u_0^{(0)}(x,y) + \tfrac{1}{2}\varepsilon(u_1^{(0)}(x,y)e^{it} + \tilde{u}_1^{(0)}(x,y)e^{-it}), \quad (33)$$

$$f^{(0)} = f_0^{(0)}(x) + \tfrac{1}{2}\varepsilon(f_0^{(0)}(x)e^{it} + \tilde{f}_0^{(0)}(x)e^{-it}), \quad (34)$$

where a tilde over a symbol represents complex conjugate, we find that the solutions satisfying (21) and (22) are

$$u_0^{(0)} = \frac{f_0^{(0)\prime}(x)}{R}\left(\frac{e^{Ry}-1}{e^R-1} - y\right) - \frac{Gr\sin\varphi}{Pr\,R^2}\left\{A_0^{(0)}e^{Ry} + B_0^{(0)} + \frac{m-1}{e^{Pr\,R}-1}\times\left[\frac{e^{Pr\,Ry}}{Pr-1} + Pr\,Ry + Pr\right] - Pr\,Ry - Pr\right\} \quad (35)$$

$$A_0^{(0)} = \frac{1}{e^R - 1}\left\{Pr\,R - \frac{m-1}{e^{Pr\,R}-1}\left[\frac{e^{Pr\,R}-1}{Pr-1} + Pr\,R\right]\right\};$$

$$B_0^{(0)} = Pr - \frac{m-1}{e^{Pr\,R}-1}\left(\frac{1}{Pr-1} + Pr\right) - A_0^{(0)}$$

and

$$u_1^{(0)} = \frac{1}{i\sigma}f_1^{(0)\prime}(x)\left\{\left(\frac{1}{e^{m_0} - e^{n_0}}\right)\times\left[(1 - e^{n_0})e^{m_0 y} + (e^{m_0} - 1)e^{n_0 y}\right] - 1\right\},$$

$$m_0, n_0 = \tfrac{1}{2}[R \pm (R^2 + 4i\sigma)^{1/2}]. \quad (36)$$

Next we determine the functions $f_{0,1}^{(0)}(x)$ by substituting (35) and (36) into the continuity equation of the order $O(\delta)$ problem, i.e. (24). We find that to satisfy the conditions on $v^{(1)}$ as given in (28) and (29) we must have

$$v^{(1)} = 0 \qquad (37)$$

and

$$f_0^{(0)''}(x) = 0 = f_1^{(0)''}(x). \qquad (38)$$

We may take

$$f_0^{(0)} = \lambda_0^{(0)} x; \quad f_1^{(0)} = \lambda_1^{(0)} x. \qquad (39)$$

Putting (34) and (39) into (32) and employing (23) we get

$$\lambda_0^{(0)} = \frac{\alpha}{K} = \lambda_1^{(0)} \qquad (40)$$

and the order $O(1)$ solution is now complete.

3.2. *Higher-order approximations*

In order to obtain higher-order approximations we substitute (31), (33) and (37) into (27) and write

$$\Theta^{(1)} = \Theta_0^{(1)}(x,y) + \tfrac{1}{2}\varepsilon(\Theta_1^{(1)} e^{it} + \tilde{\Theta}_1^{(1)} e^{-it}) + \tfrac{1}{4}\varepsilon^2(\Theta_2^{(1)} e^{2it} + \tilde{\Theta}_2^{(1)} e^{-2it}). \qquad (41)$$

We then get

$$\frac{\partial^2 \Theta_0^{(1)}}{\partial y^2} - Pr\,R\frac{\partial \Theta_0^{(1)}}{\partial y} = Pr\,\bar{E}c\left(\frac{\partial u_0^{(0)}}{\partial y}\right)^2 + \tfrac{1}{2}Pr\bar{E}c\varepsilon^2\frac{\partial u_1^{(0)}}{\partial y}\frac{\partial \tilde{u}_1^{(0)}}{\partial y},$$

$$\frac{\partial^2 \Theta_1^{(1)}}{\partial y^2} - Pr\,R\frac{\partial \Theta_1^{(1)}}{\partial y} - i\sigma\,Pr\,\Theta_1^{(1)} = 2Pr\,\bar{E}c\frac{\partial u_0^{(0)}}{\partial y}\frac{\partial u_1^{(0)}}{\partial y},$$

$$\frac{\partial^2 \Theta_2^{(1)}}{\partial y^2} - Pr\,R\frac{\partial \Theta_2^{(1)}}{\partial y} - 2i\sigma\,Pr\,\Theta_2^{(1)} = Pr\,\bar{E}c\left(\frac{\partial u_1^{(0)}}{\partial y}\right)^2.$$

With the help of (35) and (36), the solutions of these equations satisfying (28) and (29) are

$$\begin{aligned}\Theta_0^{(1)} = {} & Pr\,\bar{E}c\frac{\alpha^2}{R^2K^2}[C_{00}^{(1)} + D_{00}^{(1)} e^{Pr\,Ry} - h_{00}^{(1)}(y)] \\ & + \bar{E}c\frac{Gr^2\sin^2\varphi}{Pr\,R^3}[C_{01}^{(1)} + D_{01}^{(1)} e^{Pr\,Ry} - h_{01}^{(1)}(y)] \\ & - \bar{E}c\frac{2\alpha Gr\sin\varphi}{R^3K}[C_{02}^{(1)} + D_{02}^{(1)} e^{Pr\,Ry} - h_{02}^{(1)}(y)] \\ & - \varepsilon^2\frac{Pr\,\bar{E}c\,\alpha^2}{2\sigma^2K^2}\left|\frac{1}{e^{m_0} - e^{n_0}}\right|^2 \\ & \times [C_s^{(1)} + D_s^{(1)} e^{Pr\,Ry} - h_s^{(1)}(y)],\end{aligned} \qquad (42)$$

$$\begin{aligned}\Theta_1^{(1)} = {} & Pr\,\bar{E}c\frac{2\alpha^2}{i\sigma K^2(e^{m_0} - e^{n_0})} \\ & \times [C_{10}^{(1)} e^{m_1 y} + D_{10}^{(1)} e^{n_1 y} - h_{10}^{(1)}(y)] \\ & - \bar{E}c\frac{2\alpha\,Gr\sin\varphi}{i\sigma RK}\left(\frac{1}{e^{m_0} - e^{n_0}}\right)[C_{11}^{(1)} e^{m_1 y} \\ & + D_{11}^{(1)} e^{n_1 y} - h_{11}^{(1)}(y)],\end{aligned}$$

$$m_1, n_1 = \tfrac{1}{2}[Pr\,R \pm (Pr^2R^2 + 4i\sigma\,Pr)^{1/2}], \qquad (43)$$

and

$$\Theta_2^{(1)} = Pr\,\bar{E}c\frac{\alpha^2}{\sigma^2K^2(e^{m_0} - e^{n_0})^2}[C_{20}^{(1)} e^{m_2 y} + D_{20}^{(1)} e^{n_2 y} - h_{20}^{(1)}(y)],$$

$$m_2, n_2 = \tfrac{1}{2}[Pr\,R \pm (Pr^2R^2 + 8i\sigma\,Pr)^{1/2}]. \qquad (44)$$

For brevity we have put

$$h_{00}^{(1)}(z) = -\frac{e^{2Rz}}{2(2 - Pr)(e^R - 1)^2} + \frac{2e^{Rz}}{R(1 - Pr)(e^R - 1)} + \frac{1}{Pr^2R^2}(1 + Pr\,Rz),$$

$$\begin{aligned}h_{01}^{(1)}(z) = {} & -\frac{A_0^{(0)2} e^{2Rz}}{2R(2 - Pr)} - \frac{(m-1)^2 e^{2Pr\,Rz}}{2R(Pr - 1)^2(e^{Pr\,R} - 1)} \\ & - \frac{2A_0^{(0)}Pr}{R(1 - Pr)}\left(\frac{m-1}{e^{Pr\,R} - 1} - 1\right)e^{Rz} \\ & - \frac{2Pr(m-1)}{(Pr - 1)(e^{Pr\,R} - 1)}\left(\frac{m-1}{e^{Pr\,R} - 1} - 1\right)z\,e^{Pr\,Rz} \\ & - \frac{2\sigma(m-1)A_0^{(0)}}{R(Pr^2 - 1)(e^{Pr\,R} - 1)}e^{R(Pr+1)z} \\ & + \frac{1}{R}\left(\frac{m-1}{e^{Pr\,R} - 1} - 1\right)^2(1 + Pr\,Rz),\end{aligned}$$

$$\begin{aligned}h_{02}^{(1)}(z) = {} & -\frac{A_0^{(0)} e^{2Rz}}{(2 - Pr)(e^R - 1)} \\ & - \frac{1}{1 - Pr}\left[\frac{Pr}{e^R - 1}\left(\frac{m-1}{e^{Pr\,R} - 1} - 1\right) - \frac{A_0^{(0)}}{R}\right]e^{Rz} \\ & + \frac{m-1}{(Pr - 1)(e^{Pr\,R} - 1)}z\,e^{Pr\,Rz} \\ & - \frac{Pr(m-1)e^{R(Pr+1)z}}{(Pr^2 - 1)(e^R - 1)(e^{Pr\,R} - 1)} \\ & - \frac{1}{Pr\,R}\left(\frac{m-1}{e^{Pr\,R} - 1} - 1\right)(1 + Pr\,Rz),\end{aligned}$$

$$\begin{aligned}h_s^{(1)}(z) = {} & \frac{|m_0(1 - e^{n_0})|^2}{(m_0 + \tilde{m}_0)(m_0 + \tilde{m}_0 - Pr\,R)}e^{(m_0 + \tilde{m}_0)z} \\ & - 2\mathrm{Re}\left[\frac{m_0\tilde{n}_0(1 - e^{n_0})(1 - e^{m_0})}{(m_0 + \tilde{n}_0)(m_0 + \tilde{n}_0 - Pr\,R)}e^{(m_0 + \tilde{n}_0)z}\right] \\ & - \frac{|n_0(1 - e^{m_0})|^2}{(n_0 + \tilde{n}_0)(n_0 + \tilde{n}_0 - Pr\,R)}e^{(n_0 + \tilde{n}_0)z},\end{aligned}$$

$$h_{10}^{(1)}(z) = -\frac{m_0(1-e^{n_0})}{(e^R-1)F(R+m_0)}e^{(R+m_0)z} - \frac{n_0(e^{m_0}-1)}{(e^R-1)F(R+n_0)}e^{(R+n_0)z} + \frac{m_0(1-e^{n_0})}{RF(m_0)}e^{m_0 z} + \frac{n_0(e^{m_0}-1)}{RF(n_0)}e^{n_0 z},$$

$$h_{11}^{(1)}(z) = -\frac{A_0^{(0)}m_0(1-e^{n_0})}{F(R+m_0)}e^{(R+m_0)z} - \frac{A_0^{(0)}n_0(e^{m_0}-1)}{F(R+n_0)}e^{(R+n_0)z} - \frac{Pr\,m_0(m-1)(1-e^{n_0})}{(Pr-1)(e^{Pr\,R}-1)F(Pr\,R+m_0)}e^{(Pr\,R+m_0)z} - \frac{Pr\,R(n-1)(e^{m_0}-1)e^{(Pr\,R+n_0)z}}{(Pr-1)(e^{Pr\,R}-1)F(Pr\,R+n_0)} - \frac{Pr\,m_0(1-e^{n_0})}{F(m_0)}\left(\frac{m-1}{e^{Pr\,R}-1}-1\right)e^{m_0 z} - \frac{Pr\,n_0(e^{m_0}-1)}{F(n_0)}\left(\frac{m-1}{e^{Pr\,R}-1}-1\right)e^{n_0 z},$$

$$h_{20}^{(1)}(z) = -\frac{m_0^2(e^{n_0}-1)^2}{F_1(2m_0)}e^{2m_0 z} - \frac{n_0^2(e^{m_0}-1)^2}{F_1(2n_0)} + \frac{2m_0 n_0(e^{m_0}-1)(e^{n_0}-1)}{F_1(m_0+n_0)}e^{(m_0+n_0)z}$$

where Re stands for the real part of a complex variable, and

$F(z) = z^2 - Pr\,Rz - i\sigma Pr;$

$F_1(z) = z^2 - Pr\,Rz - 2i\sigma Pr.$

The arbitrary constants are given by

$$\begin{aligned} C_{0j}^{(1)} &= (e^{Pr\,R}h_{0j}^{(1)}(0) - h_{0j}^{(1)}(1))/(e^{Pr\,R}-1);\\ D_{0j}^{(1)} &= [h_{0j}^{(1)}(1) - h_{0j}^{(1)}(0)]/(e^{Pr\,R}-1);\\ C_{ij}^{(1)} &= [e^{n_i}h_{ij}^{(1)}(0) - h_{ij}^{(1)}(1)]/(e^{n_i}-e^{m_i});\\ D_{ij}^{(1)} &= [h_{ij}^{(1)}(1) - h_{ij}^{(1)}(0)e^{n_i}]/(e^{n_i}-e^{m_i})\\ C_s^{(1)} &= [h_s^{(1)}(0)e^{Pr\,R} - h_s^{(1)}(1)]/(e^{Pr\,R}-1),\\ D_s^{(1)} &= [h_s^{(1)}(1) - h_s^{(1)}(0)]/(e^{Pr\,R}-1). \end{aligned} \tag{45}$$

In (42), the terms independent of ε arise as a result of the steady component of the driving pressure and free convection heat transfer. The terms dependent on ε is the steady streaming component which is due to the non-linearity of the governing equations and it is this phenomenon that has attracted particular interest in purely oscillatory flows past curved bodies. The steady streaming here is due to heating in the presence of viscous dissipation. It is also present in the velocity distributions of the present approximation.

Next we calculate the pressure distribution by substituting (31) into (26) which we then solve to get

$$P^{(1)} = \frac{Gr\cos\varphi}{Pr\,R}\left[Pr\,Ry + \frac{m-1}{e^{Pr\,R}-1}(e^{Pr\,Ry} - Pr\,Ry)\right] + f^{(1)}(x,t). \tag{46}$$

Finally we express $u^{(1)}$ and $f^{(1)}(x,t)$ also in the form

$$u^{(1)} = u_0^{(1)} + \tfrac{1}{2}\varepsilon(u_1^{(1)}e^{it} + \bar{u}_1^{(1)}e^{-it}) + \tfrac{1}{4}\varepsilon^2(u_2^{(1)}e^{2it} + \bar{u}_2^{(1)}e^{-2it}),$$

$$f^{(1)} = f_0^{(1)}(x) + \tfrac{1}{2}\varepsilon(f_1^{(1)}e^{it} + \bar{f}_1^{(1)}e^{-it}) + \tfrac{1}{4}\varepsilon^2(f_2^{(1)}e^{2it} + \bar{f}_2^{(1)}e^{-2it}),$$

then (25) similarly separates out as

$$\frac{\partial^2 u_0^{(1)}}{\partial y^2} - R\frac{\partial u_0^{(1)}}{\partial y} = f_0^{(1)\prime} - Gr\sin\varphi\,\Theta_0^{(1)},$$

$$\frac{\partial^2 u_1^{(1)}}{\partial y^2} - R\frac{\partial u_1^{(1)}}{\partial y} - i\sigma u_1^{(1)} = f_1^{(1)\prime} - Gr\sin\varphi\,\Theta_1^{(1)},$$

$$\frac{\partial^2 u_2^{(1)}}{\partial y^2} - R\frac{\partial u_2^{(1)}}{\partial y} - 2i\sigma u_2^{(1)} = f_2^{(1)\prime} - Gr\sin\varphi\,\Theta_2^{(1)}.$$

Hence the solution of these equations can be written as

$$u_j^{(1)}(x,y) = f_j^{(1)\prime}(x)H_j^{(1)}(y) - Gr\sin\varphi\,G_j^{(1)}(y),$$

and if these are substituted into the relevant equations for the order $O(\delta^2)$ problem, namely

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} = 0,\quad v^{(2)}|_{y=0} = 0 = v^{(2)}|_{y=1},$$

then we find that

$$v^{(2)} = 0, \tag{47}$$

$$f_1^{(0)\prime\prime}(x) = 0 = f_1^{(1)\prime\prime}(x) = f_2^{(1)\prime\prime}(x). \tag{48}$$

As a result of (30) we have

$$f_0^{(1)}(x) = 0 = f_1^{(1)}(x) = f_2^{(1)}(x). \tag{49}$$

So we can employ (42)–(44) and derive the solutions

$$\begin{aligned} u_0^{(1)} = &- Pr\,\bar{E}c\frac{\alpha^2 Gr\sin\varphi}{R^2K^2}[A_{00}^{(1)} + B_{00}^{(1)}e^{Ry} - g_{00}^{(1)}(y)]\\ &- \bar{E}c\frac{Gr^2\sin^3\varphi}{Pr\,R^3}[A_{01}^{(1)} + B_{01}^{(1)}e^{Ry} - g_{01}^{(1)}(y)]\\ &+ \bar{E}c\frac{2\alpha\,Gr\sin^3\varphi}{R^3K}[A_{02}^{(1)} + B_{02}^{(1)}e^{Ry} - g_{02}^{(1)}(y)]\\ &- \varepsilon^2\frac{Pr\,\bar{E}c\,Gr\,\alpha^2\sin\varphi}{2\sigma^2K^2}\left|\frac{1}{e^{m_0}-e^{n_0}}\right|^2\\ &\times[A_s^{(1)} + B_s^{(1)}e^{Ry} - g_s^{(1)}(y)], \end{aligned} \tag{50}$$

$$u_1^{(1)} = -Pr\,\bar{E}c\frac{2\alpha^2 Gr\sin\varphi}{i\sigma K(e^{m_0}-e^{n_0})} \times[A_{10}^{(1)}e^{m_1^* y} + B_{10}^{(1)}e^{n_1^* y} - g_{10}^{(1)}(y)]$$

$$+\bar{E}c\frac{2\alpha Gr^2\sin^2\varphi}{i\sigma KR}[A_{11}^{(1)}e^{m_1^* y}+B_{11}^{(1)}e^{n_1^* y}-g_{11}^{(1)}(y)]$$

$$m_1^*, n_1^* = \tfrac{1}{2}[R \pm (R^2+4i\sigma)^{1/2}], \tag{51}$$

$$u_2^{(1)} = -Pr\,\bar{E}c\frac{Gr\,\alpha^2\sin\varphi}{\sigma^2K^2(e^{m_0}-e^{n_0})^2}\times[A_{20}^{(1)}e^{m_2^* y}+B_{20}^{(1)}e^{n_2^* y}-g_{20}^{(1)}(y)],$$

$$m_2^*, n_2^* = \tfrac{1}{2}[R+(R^2+8i\sigma)^{1/2}]. \tag{52}$$

Now we have set

$$g_{0j}(z) = e^{Rz}\int^z e^{-Rz}dz\int^z h_{0j}^{(1)}(z)dz + C_{0j}^{(1)}\frac{1}{R^2}(1+Rz) - D_{0j}^{(1)}\frac{e^{Pr\,Rz}}{R^2Pr(Pr-1)},$$

$$g_s^{(1)}(z) = e^{Rz}\int^z e^{-Rz}dz\int^z h_s^{(1)}(z)dz + C_s^{(1)}\frac{1}{R^2}(1+Rz) - D_s^{(1)}\frac{e^{Pr\,Rz}}{R^2Pr(Pr-1)},$$

$$g_{10}^{(1)}(z) = -\frac{C_{10}^{(1)}}{G(m_1)}e^{m_1z} - \frac{D_{10}^{(1)}}{G(n_1)}e^{n_1z} - \frac{m_0(1-e^{n_0})e^{(R+m)z}}{(e^R-1)F(R+m_0)G(R+m_0)} - \frac{n_0(e^{m_0}-1)e^{(R+n_0)z}}{(e^R-1)F(R+n_0)G(R+n_0)} + \frac{m_0(1+e^{n_0})e^{m_0z}}{RF(m_0)G(m_0)} + \frac{n_0(e^{m_0}-1)e^{n_0z}}{RF(n_0)G(n_0)},$$

$$g_{11}^{(1)}(z) = -\frac{C_{11}^{(1)}}{G(m_1)}e^{m_1z} - \frac{D_{11}^{(1)}}{G(n_1)}e^{n_1z} - \frac{A_0^{(0)}m_0(1-e^{n_0})}{F(R+m_0)G(R+n_0)}e^{(R+m_0)z} - \frac{A_0^{(0)}n_0(e^{m_0}-1)}{F(R+n_0)G(R+n_0)}e^{(R+n_0)z}$$
$$- \frac{Pr\,m_0(m-1)(1-e^{n_0})e^{(Pr\,R+m_0)z}}{(Pr-1)(e^{Pr\,R}-1)F(Pr\,R+m_0)G(Pr\,R+n_0)} - \frac{Pr\,n_0(m-1)(e^{m_0}-1)e^{(Pr\,R+n_0)z}}{(Pr-1)(e^{Pr\,R}-1)F(Pr\,R+n_0)G(Pr\,R+n_0)}$$
$$- \frac{Pr\,m_0(1-e^{n_0})}{F(m_0)G(m_0)}\left(\frac{m-1}{e^{Pr\,R}-1}-1\right)e^{m_0z} - \frac{Pr\,n_0(e^{m_0}-1)}{F(n_0)G(n_0)}\left(\frac{m-1}{e^{Pr\,R}-1}-1\right)e^{n_0z},$$

$$g_{20}^{(1)}(z) = -\frac{C_{20}}{G_1(m_2)}e^{m_2z} - \frac{D_{20}^{(1)}}{G_1(n_2)}e^{n_2z} - \frac{m_0^2(e^{n_0}-1)^2e^{2m_0z}}{F_1(2m_0)G_1(2m_0)} - \frac{n_0^2(e^{m_0}-1)^2e^{2n_0z}}{F_1(2n_0)G_1(2n_0)} + \frac{2m_0n_0(e^{m_0}-1)(e^{n_0}-1)}{F_1(m_0+n_0)G_1(m_0+n_0)}e^{(m_0+n_0)z}$$

where

$$G(z) = z^2 - Rz - i\sigma, \quad G_1(z) = z^2 - Rz - 2i\sigma.$$

The arbitrary constants $A_{00}^{(1)}$, $B_{00}^{(1)}$ etc. can be written as in (45). The solution to order $O(\delta)$ is now complete.

First of all we observe that the transverse velocity is the same as for the idealized infinite channel. For the horizontal channel ($\varphi = 0$) the axial velocity becomes exact and is independent of δ, its value being given by

$$u = u_0^{(0)} + \tfrac{1}{2}\varepsilon(u_1^{(0)}e^{it} + \bar{u}_1^{(0)}e^{-it}), \tag{53}$$

where $u_0^{(0)}$ and $u_1^{(0)}$ are given by (35) and (36) with $\varphi = 0$. It is therefore independent of the free convection parameter Gr and corresponds essentially to the solution for an infinite channel. When the channel is vertical the axial velocity exhibits a dependence upon Gr for all approximations.

Even in the case of a horizontal channel, the pressure distribution shows a strong dependence upon y as a result of the free convection heat transfer. The only exception is for a vertical channel where it is given simply by

$$P = \frac{\alpha}{K}x(1+\cos t)\cdot\frac{1}{\delta}\cdot$$

Unlike the velocity distribution, the temperature distribution is always δ-dependent for all channel orientations and even in the limit as $Gr \to 0$. Therefore the effect of finiteness of the channel length has more profound effect on the temperature and pressure of the fluid.

4. SHEAR STRESSES AT THE WALLS

From the physical point of view, it is necessary to know the shear stress at the walls of the channel. They are given by

$$\tau_{0,1}^1 = \mu\left.\frac{\partial u'}{\partial y'}\right|_{y'=0,h}$$

which by virtue of (9) can be written as

$$\tau_{0,1} = \frac{h\tau_{0,1}^1}{\mu V_0} = \left.\frac{\partial u}{\partial y}\right|_{y=0,1} = D_{0r,1r} + iD_{0i,1i} = |D_{0,1}|e^{i\alpha_{0,1}}. \tag{54}$$

Since the velocity profile is identical to that of an infinite channel when $\varphi = 0$, we consider the situation for a vertical channel.

The values for shear stresses for $\varphi = \pi/2$ are entered in Table 1 for various values of the parameter. First of

Table 1

G	$\bar{E}c$	m		D_0	D_1	$\tan\alpha_0$	$\tan\alpha_1$
5.0	0.1	−1.0	1	5.9108	2843.4753	0.2426	−0.0864
			5	6.0875	2843.7578	0.2254	−0.4197
			10	6.0329	2844.2840	−0.1490	−0.7698
			15	5.7365	2854.7101	−0.5107	−1.0131
5.0	0.1	2.0	5	6.1384	−479.5440	0.2354	−0.4297
			10	6.0838	−479.0178	−0.1490	−0.7698
5.0	0.2	2.0	5	5.7752	−948.9680	0.2354	−0.4297
10.0	0.2	2.0	5	9.8240	−1500.0000	0.2754	−0.4997

$R = 1.0$, $Pr = 0.71$, $\alpha = 1.0$, $\delta = 0.01$, $\xi = 1$, $t = \pi$.

Table 2

G	$\bar{E}c$	m		E_0	E_1	$\tan\alpha_0$	$\tan\alpha_1$
5.0	0.1	−1.0	1	−0.4165	17.0535	0.3473	2.7389
			5	−0.8088	8.3545	0.5938	1.3015
			10	−0.9805	5.2226	0.6679	0.5116
			15	−0.8404	1.1221	1.0153	2.1358
5.0	0.1	2.0	5	0.9713	5.0192	−0.0367	0.2202
			10	0.9488	4.8514	1.3399	−0.6019
5.0	0.2	2.0	5	1.2759	9.3518	−0.0367	0.2202
10.0	0.2	2.0	5	1.4711	14.8436	−0.4322	0.3806

$R = 1.0$, $Pr = 0.71$, $\alpha = 1.0$, $\delta = 0.01$, $\xi = 1$, $t = \pi$.

all it can be observed that the shear at the upper wall is much larger than that at the lower wall. When the average of the temperatures of the two walls is equal to that of the static fluid ($m = -1$), the shear stresses are positive; but for unequal wall temperatures ($m = 2$) the shear stress on the upper wall changes direction. For the lower wall an increase in frequency causes a rise in the shear stress at low frequencies and decrease at high frequencies. The shear stress at the upper wall always increases with increase in frequency. A rise in the free convection parameter Gr results in corresponding rise in the shear stress.

However an increase in the viscous dissipation heat $\bar{E}c$ causes a reduction in the stress on the lower wall and rise on the upper wall.

5. THE HEAT FLUX

Knowing the temperature distribution we can calculate the heat transfer, q', at the walls from the relation

$$q' = -k\frac{\partial T'}{\partial y'}\bigg|_{y'=0,h}$$

which by virtue of (2.5) reduces to the following non-dimensional form:

$$q = -\frac{hq'}{k(T_0 - T_\infty)} = \frac{\partial\Theta}{\partial y}\bigg|_{y=0,1} = |E_{0,1}|\,e^{i\beta_{0,1}}. \quad (55)$$

Again when $\varphi = \pi/2$, the values of (55) are entered in Table 2.

Similarly it is observed that the heat transfer rates are higher at the upper wall than the lower wall. When the frequency increases the heat transfer at the lower wall increases (in absolute terms) for small frequencies but decreases at the higher frequencies. The heat flux at the upper wall decreases always with increase in frequency. Also the heat flux at the lower wall changes sign when m changes from -1 to 2. An increase in viscous dissipation heat $\bar{E}c$ and free convection parameter Gr causes a corresponding increase in the heat flux at the walls.

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ECOULEMENT PULSATOIRE DANS UN CANAL POREUX ET CHAUFFE

Résumé—On étudie la convection mixte dans un écoulement à travers un canal poreux quand une pression pulsatoire est appliquée aux extrémités. On suppose que le rapport de la largeur du canal à la longueur (δ) est petit. Dans cette situation physiquement réaliste d'un canal fini, la vitesse transversale demeure non perturbée et elle acquiert sa valeur de succion ou d'injection aux parois du canal. Un point saillant de l'étude est la présence d'une composante stationnaire dans les approximations d'ordre élevé due à la non-linéarité dans le terme de dissipation visqueuse. Quand le canal est horizontal, la distribution de vitesse axiale obtenue est exacte et indépendante de δ. Mais les distributions de pression et de température sont exactes et dépendantes de δ. Par ailleurs, si le canal est vertical, seule est obtenue exactement la distribution de pression. Elle est indépendante de la coordonnée transversale mais elle varie linéairement avec la coordonnée axiale. La vitesse et la température sont obtenues approchées à l'ordre $O(\delta)$. La tension de cisaillement et le flux de chaleur aux parois sont discutés quantitativement.

PULSIERENDE STRÖMUNG IN EINEM BEHEIZTEN PORÖSEN KANAL

Zusammenfassung—Die erzwungene Strömung mit überlagerter freier Konvektion in einem porösen Kanal, an dessen Enden ein pulsierender Druck aufgeprägt ist, wird erörtert. Dabei wird angenommen, daß das Verhältnis von Kanalbreite zu -länge (δ) klein ist. Nur in diesem physikalisch realistischen Fall eines begrenzten Kanals bleibt die Quer-Geschwindigkeit ungestört und erreicht ihren Saug-oder Ausblaseffekt an den Kanalwänden. Ein herausragendes Ergebnis der Untersuchung ist das Auftreten eines stetigen Anteils der Strömung bei den Approximationen höherer Ordnung, verursacht durch eine Nichtlinearität der viskosen Dissipation. Bei waagerechtem Kanal ist die berechnete Verteilung der Axialgeschwindigkeit exakt und unabhängig von δ. Druck- und Temperaturverteilung sind jedoch exakt und δ-abhängig. Andererseits ergibt sich bei senkrechtem Kanal nur die Druckverteilung exakt. Sie ist unabhängig von der Quer-Koordinate, ändert sich jedoch linear mit der Längs-Koordinate. Geschwindigkeit und Temperatur ergeben sich näherungsweise als Funktion von δ. Ebenso werden Schubspannung und Wärmestromdichte an der Wand quantitativ beschrieben.

ПУЛЬСАЦИОННОЕ ТЕЧЕНИЕ В НАГРЕВАЕМОМ ПОРИСТОМ КАНАЛЕ

Аннотация — Рассматривается смешанное течение при свободной и вынужденной конвекции в пористом канале в случае, когда пульсирующее давление приложено поперек его концов. Предполагается, что отношение ширины канала к его длине (δ) является малой величиной. Показано, что в этой физически реальной ситуации канала конечной длины поперечная компонента скорости остается неизменной и принимает значение скорости вдува или отсоса на стенке. Отличительной особенностью задачи является наличие стационарной компоненты скорости течения в аппроксимациях высшего порядка из-за нелинейности вязкой диссипации тепла. В случае, когда канал расположен горизонтально, можно найти точное распределение аксиальной скорости, которое не зависит от величины параметра δ. В то же время распределения давления и температуры являются точными и зависящими от параметра δ. С другой стороны, если канал расположен вертикально, то точной величиной является только распределение давления, которое не зависит от поперечной координаты и изменяется линейно с осевой координатой. Значения скорости и температуры определены приближенно с точностью до $O(\delta)$. Найдены также напряжение сдвига и плотность теплового потока на стенке.